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Isoparametric hypersurfaces and moment map

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Abstract. The classification of isoparamtric hypersurfaces has been almost completed in these five years. The discovery of infinitely many non-homogeneous isoparametric hypersurfaces in S^n by Ozeki-Takeuchi stimulates Ferus-Karcher-Münzner to obtain a celebrated result to the effect that one can construct isoparametric hypersurfaces from all the representations of Clifford algebras. The polynomials defining these hypersurfaces are expressed by the moment map of the spin action, which is the main theme of this article.

1 Introduction

Curves filling a plane in a tidy way are parallel lines or parallel circles. The 3-space is filled similarly by parallel planes, parallel spheres or parallel right circular cylinders. We are concerned with:

Q. Which surface M has self-similar parallel surfaces?
Which surface has parallel surfaces which are all regular?

Here we mean by a parallel surface the surface located in the same distance from the original one in the normal direction. In general, a singularity occurs when we reach a focal point. Therefore, "parallel surfaces are all regular" means a focal subset is also regular.

This problem has its origin in the geometric optics and the analysis of wave fronts developing by the Huygens principle.

A. To the same problem for hypersurfaces in E^n, the answer is hyperplanes, hyperspheres and the product of these. In the hyperbolic space H^n, they are equidistant hypersurfaces, horospheres, spheres and the product of these (É. Cartan). However,

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in the sphere S^n, infinitely many different kinds of parallel hypersurfaces with this property exist.

Q. How do we express M?

A. Level set expression: Expression of M as a level set $M = f^{-1}(t)$ of a global function f on the ambient space is suitable to express developing wave fronts. In the research of mean curvature flow, this method is common.

Remark 1.1. *Functions expressing M are not unique.*

- *$f(x) = \|x\|$ and $g(x) = \cos\|x\|$ have the same level sets (round spheres).*

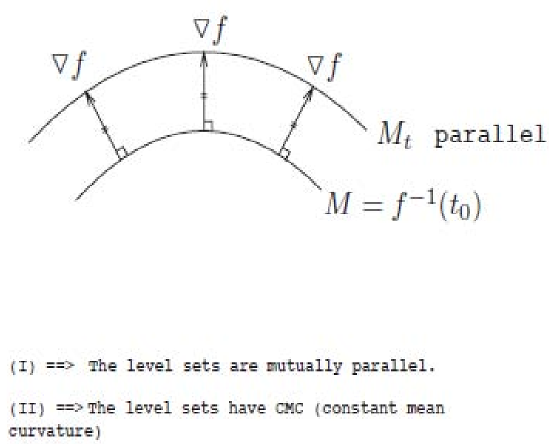
Now, let $\overline{M}$ be a complete Riemannian manifold, ∇ the Levi-Civita connection, and $\triangle$ the Laplacian.

Definition 1.2. (1) A C^2 function $f : \overline{M} \to \mathbb{R}$ is called an **isoparametric function** if f satisfies:

(I) $\|\nabla f\|^2 = \varphi(f), \quad \varphi : f(\overline{M}) \to \mathbb{R} : C^2$

(II) $\triangle f = \psi(f), \quad \psi : f(\overline{M}) \to \mathbb{R} : C^0$

(2) We call a level set of a regular value of f an **isoprametric hypersurface**.



Example 1. $f(x) = \|x\|^2$ has $\nabla f = 2x$ and so $\|\nabla f\|^2 = 4f$, satisfying $\Delta f = 2n$, and hence f is an isoparametric function.

Fact 1. (É. Cartan) *Let $\overline{M}$ be the space forms (E^n, S^n or $\mathbb{H}^n$), and consider a family of parallel hypersurfaces $\{M_t\}$. Then the following are equivalent:*

(i) *$\{M_t\}$ is a family of isoparametric hypersurfaces.*

(ii) *All M_t has constant mean curvature.*

(iii) *Certain M_t has constant principal curvatures.*

Remark 1.3. *A local notion(iii) induces a global notion (i).*

Planes, spheres and right cylinders have constant principal curvatures, and so isoparametric.

Known examples

$\overline{M}$	M^{n-1}		
E^n	E^{n-1} or S^{n-1}	$E^k \times S^{n-k-1}$	–
H^n	H_{eq} or S^{n-1}	$H^k_{eq} \times S^{n-k-1}$	–
S^n	S^{n-1}	$S^k \times S^{n-k-1}$	more

H_{eq}: equidistant hypersurfaces and horosphers

{homogeneous hypersurfaces} $\subset$ {isoparametric hypersurfaces} follows immediately from **Fact 1**.

Fact 2. (É. Cartan, '37)

- When $\overline{M} = E^n$ or H^n, the equality holds.
- In S^n, there exists homogeneous examples more than hyperspheres and product of spheres.

Fact 3. (Ozeki-Takeuchi, '76) *There exist infinitely many non-homogeneous isoparametric hypersurfaces in S^n.*

In the following, we restrict our argument to $\overline{M} = S^n$.

Remark 1.4. *Homogeneous hypersurfaces in S^n are given by isotropy orbits of rank two symmetric spaces, and classified completely (Hsiang-Lawson, '71). Homogeneous hypersurfaces are characterized as a completely integrable system (Ferapontov, '95).*

Fact 4. (Münzner, '81) *An isoparametric hypersurface M_t in S^n satisfies the following :*

(a) $g = \sharp\{\text{distinct principal curvatures}\} \in \{1, 2, 3, 4, 6\}$.

(b) The multiplicities $m_1, m_2, \dots, m_g$ of the principal curvatures $\lambda_1 > \lambda_2 > \cdots > \lambda_g$ satisfy $m_i = m_{i+2}$.

(c) There exists a degree g homogeneous polynomial $F : E^{n+1} \to \mathbb{R}$ called the **Cartan-Münzner polynomial** satisfying :

(i) $\|DF(x)\|^2 = g^2\|x\|^{2g-2}$

(ii) $\triangle F(x) = \dfrac{m_2 - m_1}{2} g^2 \|x\|^{g-2}$.

Then $f = F|_{S^n} : S^n \to [-1, 1]$ is an isoparametric function on S^n. The level set $M_t = F^{-1}(t) \cap S^n$ for $t \in (-1, 1)$ is an isoparametric hypersurface.

Definition 1.5. We call $M_\pm = f^{-1}(\pm 1)$ the **focal submanifold**.

Classification of isoparametric hypersurfaces in S^n

g	1	2	3	4 (partial)	6
M	S^{n-1} hom.	$S^k \times S^{n-k-1}$ hom.	$C_\mathbb{F}$ hom.	hom. or OT-FKM type	N^6, M^{12} hom.

<u>**The case** $g = 3$</u>: Cartan hypersurface $C_\mathbb{F}^{3d}$

Fact 5. (Cartan '38) *Isoparametic hypersrufaces $C_\mathbb{F}^{3d}$ with $g = 3$ are given by tubes over $P^2\mathbb{F}$ which are standardly embedded in S^4, S^7, S^{13}, S^{25}. Here, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathcal{C}$ (Cayley number). ($d = 1, 2, 4, 8 = m = m_i$).*

<u>**The case** $g = 6$**:**</u>

Fact 6. (Abresch, '83) *When $g = 6$ we have $m_i = m \in \{1, 2\}$.*

For each m, we have a homogeneous example:

$m = 1$: Isotropy orbits of $G_2/SO(4)$ in S^7.

$m = 2$: Isotropy orbits of $G_2 \times G_2/G_2$ in S^{13}.

These hypersurfaces are closely related to Cartan hypersurfaces $C_\mathbb{F}$ as follows:

Proposition 1. (M. '93) *The homogeneous hypersurface N^6 with $(g, m) = (6, 1)$ has a fibration $\pi : N^6 \to S^3$ with fiber $C_\mathbb{R}$ (in fact, $N^6 \cong S^3 \times C_\mathbb{R}$ holds).*

Proposition 2. (M. '11) *The homogeneous hypersurface M^{12} with $(g, m) = (6, 2)$ has a fibration $\pi : M \to S^6$ with fiber $C_\mathbb{C}$.*

$m = 1$

$N^6 \cong SO(4)/Z_2 \oplus Z_2$
$\downarrow \leftarrow C_\mathbb{R} \cong SO(3)/Z_2 \oplus Z_2$
$S^3 \cong SO(4)/SO(3)$

$m = 2$

$M^{12} \cong G_2/T^2$
$\downarrow \leftarrow C_\mathbb{C} \cong SU(3)/T^2$
$S^6 \cong G_2/SU(3)$

Remark 1.6. *The focal submanifolds $M_{\pm}$ with $(g,m) = (6,2)$ are related to Bryant's twistor fibration:*

(i) $M_+ \cong \mathbb{Q}^5 \to S^6 = G_2/SU(3)$ *is diffeomorphic to the twistor fibration on* S^6 *with fiber* $\mathbb{C}P^2$.

(ii) $M_- \cong \mathbb{Q}^5 \to G_2/SO(4)$ *is diffeomorphic to the twistor fibration on* $G_2/SO(4)$ *with fiber* $\mathbb{C}P^1$.

- *Thus* $M+$ *and* M_- *are not congruent.*

Yau's problem: Classify isoparametric hypersurfaces in S^n(1992).

Fact 7. (Dorfmeister-Neher, '85, M. '09) *Isoparametric hypersurfaces with* $(g,m) = (6,1)$ *are given by isotropy orbits of* $G_2/SO(4)$.

Theorem 1. (M. to appear in Ann. of Math.) *Isoparametric hypersurfaces with* $(g,m) = (6,2)$ *are homogeneous, i.e., isotopy orbits of* $G_2 \times G_2/G_2$.

Key Proposition 3. (M. '93, '11) *An isoparametric hypersurface with* $g = 6$ *is homogeneous* ⇔ **Condition A**; *The kernel of the shape operator of a focal submanifold is independent of the normal direction.*

(When $m = 2$, to show **Condition A** is extremely difficult.)

The case $g = 4$:

Fact 8. (Cecil-Chi-Jensen, Immervoll, Chi, '07∼'12) *Isoparametric hypersurfaces with* $g = 4$ *are exhausted by the following table, except for* $(m_1, m_2) = (7, 8)$.

isoparametric hypersurfaces with $g = 4$ in S^n.

<table>
<tr><td rowspan="3">OT-FKM type</td><td>non-homogeneous</td><td>$(m_1, m_2) = (3, 4k), (7, 8k), \ldots$</td></tr>
<tr><td rowspan="4">hom.:
isotropy orbits
of G/K</td><td>G/K : non-Hermitian
$(4, 4k-1)$</td></tr>
<tr><td>*Hermitian
$(1, k), (2, 2k-1), (9, 6)$</td></tr>
<tr><td rowspan="2">not OT-FKM type</td><td>*Hermitian $(4, 5)$</td></tr>
<tr><td>non-Hermitian $(2, 2)$</td></tr>
</table>

2 Clifford systems and OT-FKM type

Let $O(n)$ be a orthogonal group, $\mathfrak{o}(n)$ be its Lie algebra.

Definition 2.1. $P_0, \dots P_m \in O(2l)$ is a **Clifford system**

$$\Leftrightarrow \boxed{P_iP_j + P_jP_i = 2\delta_{ij}\mathrm{id}, \quad 0 \le i, j \le m.}$$

Remark 2.2. (1) *Possible pairs* (m, l)*: (Atiyah-Bott-Shapiro, '64)*

m	1	2	3	4	5	6	7	8	$\cdots$	$m+8$	$\cdots$
$l = \delta(m)$	1	2	4	4	8	8	8	8	$\cdots$	$16\delta(m)$	$\cdots$

(2) *With respect to the* **inner product**

$$\langle P, Q\rangle = \frac{1}{2l}\mathrm{Tr}(P^tQ),$$

$P_0, \dots, P_m$ *is an orthonormal basis of the space* V *spanned by themselves.*

(3) *Clifford system corresponds to an expression of Clifford algebra in a one-to-one way.*

(4) *Each* P_i *is a symmetric orthogonal matrix.*

(5) P_iP_j *are skew-symmetric.*

Fact 9. (Ferus-Karcher-Münzner '81) *For a Clifford system* $P_0, \dots, P_m$*,*

$$F(x) = \langle x, x\rangle^2 - 2\sum_{i=0}^{m} \langle P_i x, x\rangle^2$$

is a degree four Cartan-Münzner polynomial. When $l - m - 1 > 0$*, a level set of a regular value of* $F|_{S^{2l-1}}$ *is an isoparametric hypersurface in* S^{2l-1} *with* $g = 4$*,* $m_1 = m$*,* $m_2 = l - m - 1$*.*

By (5), P_iP_j, $0 \le i < j \le m$ are skew-symmetric, and generate a Lie subalgebra isomorphic to $\mathfrak{o}(m+1)$ in $\mathfrak{o}(2l)$.

Fact 10. (FKM, '81, p.496) *Spin*$(m+1)$ *acts on* $\mathbb{R}^{2l}$*, and preserves* $F(x)$*, namely,* $F(x)$ *is constant on each Spin*$(m+1)$ *orbit.*

Remark 2.3. *Spin*$(m+1)$ *action is small in general, and not transitive on a hypersurface.*

Goal: Express $F(x)$ via the moment map of the spin action.

3 Symplectic geometry and the main results

Definition 3.1. (1) (P^{2n}, ω) is a **symplectic manifold** $\Leftrightarrow$ ω is a non-degenerate closed 2-form on P.

(2) For $f \in \mathbb{C}^\infty(P)$, the **Hamiltonian vector field H_f** $\Leftrightarrow$ $df = \omega(H_f, \ )$.

Put $\mathrm{Ham}(P) = \{H_f \mid f \in \mathbb{C}^\infty(P)\}$. Let K be a Lie group acting on P, and $\mathfrak{k}$ be its Lie algebra.

Definition 3.2. (1) **Fundamental vector field on P** $\Leftrightarrow$ vector field given by $X_\zeta = \left.\dfrac{d}{dt}\right|_{t=0} (\exp t\zeta)x$ for $\zeta \in \mathfrak{k}$.

(2) $K \curvearrowright P$ is a **symplectic action** $\Leftrightarrow$ for $k \in K$ holds $k^*\omega = \omega$.

(3) $K \curvearrowright P$ is a **Hamiltonian action** $\Leftrightarrow$ for $\zeta \in \mathfrak{k}$, $X_\zeta \in \mathrm{Ham}(P)$, namely, there exists $\mu_\zeta \in \mathbb{C}^\infty(P)$ satisfying $d\mu_\zeta = \omega(X_\zeta, \ )$.

(4) With respect to the coadjoint action of K on $\mathfrak{k}^*$, $\mu : P \to \mathfrak{k}^*$ is the **moment map**

$$\Leftrightarrow \begin{array}{l} \text{(i) } \mu \text{ is } K \text{ equivariant} \\ \text{(ii) } d\mu(\zeta) = \omega(X_\zeta, \ ) \end{array}$$

Therefore, $K \curvearrowright P$ is a Hamiltonian action $\Leftrightarrow$ there exists moment map $\mu : P \to \mathfrak{k}^*$.

In fact, if the moment map $\mu : P \to \mathfrak{k}^*$ exists, for $\zeta \in \mathfrak{k}$, $\mu_\zeta(p) = \mu(p)(\zeta) \in C^\infty(P)$ and so $H_{\mu_\zeta} = X_\zeta$. The converse is easy.

Example 2. (1) $(\mathbb{C}^n, J, \omega)$ is a symplectic manifold with $\omega(X, \ ) = -\langle JX, \ \rangle$. When $K \curvearrowright \mathbb{C}^n$ is a Hamiltonian aciton, the moment map is give by $\mu(z)(\zeta) = -\dfrac{1}{2}\langle JX_\zeta, z\rangle$. In fact, we can show $d\mu(Y_z)(\zeta) = -\langle JX_\zeta, Y\rangle$ from $J\zeta z = \zeta Jz$. On the other hand, we have $\omega(X_\zeta, Y) = -\langle JX_\zeta, Y\rangle$.

(2) Let G/K be a Hermitian symmetric space, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. By an element $\mathfrak{z}$ of the non-trivial center $\mathfrak{c}$ of $\mathfrak{k}$, the Kähler structure J on $\mathfrak{p}$ is given by

$$Jx = \mathrm{ad}\mathfrak{z}(x) = -\mathrm{ad}x(\mathfrak{z}), \quad \mathfrak{z} \in \mathfrak{c},\ x \in \mathfrak{p}.$$

Then the isotropy action $K \curvearrowright \mathfrak{p}$ is Hamiltonian, and the **moment map** is given by

$$\mu^H(x) = \frac{1}{2}(\mathrm{ad}x)^2\mathfrak{z}$$

(Ohnita, '05).

Remark 3.3. *There <u>does not necessarily exist</u> symplectic structures on a general symmetric space.*

<u>Symplectic structure on $T\mathbb{R}^n$</u>

Because of $T\mathbb{R}^n \cong \mathbb{C}^n$, the symplectic structure is induced from $\mathbb{C}^n$.

<u>Hamiltonian action on $T\mathbb{R}^n$</u>

When $K \subset O(n)$ acts on $\mathbb{R}^n$, we extend it naturally to an action $K \curvearrowright T\mathbb{R}^n$. For $\zeta \in \mathfrak{o}(n)$, we have $X_\zeta = (\zeta x, \zeta Y)$, $(x, Y) \in T\mathbb{R}^n$.

Proposition 4. $K \curvearrowright T\mathbb{R}^n$ is a Hamiltonian action, and the moment map $\mu : T\mathbb{R}^n \to \mathfrak{k}^*$ is given by :

$$\mu(x, Y)(\zeta) = -\langle \zeta x, Y \rangle.$$

Example 3. When $n = 3$, let $\zeta_1, \zeta_2, \zeta_3 \in \mathfrak{o}(3)$ be an orthonormal frame, then for $(x, Y) \in T\mathbb{R}^3$, $\mu(x, Y)(\zeta_i) = -\langle \zeta_i x, Y \rangle$ is the well-known **angle momentum**. In particular, the moment map is given by

$$\mu(x, Y) = -\sum_{i=1}^{3} \langle \zeta_i x, Y \rangle \zeta_i.$$

<u>$Spin(m+1)$ **action on** $T\mathbb{R}^{2l}$</u>

Let $P_0, \ldots, P_m$ be a Clifford system on $\mathbb{R}^{2l}$. Then $\zeta_{ij} = P_iP_j \in \mathfrak{o}(2l)$, $0 \leq i < j \leq m$ act on $\mathbb{R}^{2l}$ and generate $\mathfrak{o}(m+1)$. Apply above argument to the $Spin(m+1)$ action $(\exp tP_iP_j)x$ on $\mathbb{R}^{2l}$. Since $\zeta_{ij} = P_iP_j$ is an orthonormal basis of $\mathfrak{o}(m+1)$, we obtain:

Proposition 5. (M.) The moment map of the $Spin(m+1)$ action on $T\mathbb{R}^{2l}$ is given by

$$\mu(x, Y) = -\sum_{0 \leq i < j \leq m} \langle \zeta_{ij} x, Y \rangle \zeta_{ij} \in \mathfrak{o}(m+1) \cong \mathfrak{o}^*(m+1).$$

In particular, we have $\|\mu(x,Y)\|^2 = \sum_{0\le i<j\le m}\langle P_iP_jx, Y\rangle^2$.

The action $U(1) \curvearrowright T\mathbb{R}^{2l}$ associated to the complex structure J commutes with J, and so a symplectic and Hamiltonian action.

Theorem 2. (M, to appear in Math. Ann.)
Let $P_0,\dots,P_m$ on $\mathbb{R}^{2l}$ be a Clifford system, and define a vector field $Y : \mathbb{R}^{2l} \to T\mathbb{R}^{2l}$: (not necessarily continuous) by

$$Y_x = \begin{cases} P_0x, & \text{if } \langle P_0x,x\rangle = 0 \\ \dfrac{\langle P_1x,x\rangle P_0x - \langle P_0x,x\rangle P_1x}{\sqrt{\langle P_1x,x\rangle^2 + \langle P_0x,x\rangle^2}}, & \text{if } \langle P_0x,x\rangle \neq 0. \end{cases}$$

Let $\mu_0 + \mu$ be the moment map of the action

$$U(1) \times Spin(m+1) \curvearrowright T\mathbb{R}^{2l}.$$

Then the Cartan-Münzner polynomial is expressed as

$$F(x) = \|\mu_0(x,Y_x)\|^2 - 2\|\mu(x,Y_x)\|^2.$$

Remark 3.4. (1) *The right hand side of $F(x)$ is determined by $x \in \mathbb{R}^{2l}$.*

(2) *We can replace P_0, P_1 by arbitrary orthogonal two unit elements in V.*

(3) *$C = \{(x, Y_x) \in T\mathbb{R}^{2l}\}$ is a $2l$ dimensional submanifold of $T\mathbb{R}^{2l}$ outside $\{x \mid \langle P_0x,x\rangle = 0\}$, but is not Lagrangian.*

(4) *For isotropy orbits of a Hermitian symmetric space, an expression via the moment map was first given by S. Fujii (2011), and Fujii-H. Tamaru.*

Remaining case

There are two non-OT-FKM type isoparametric hypersurfaces.

A brief review of homogeneous hypersurfaces

Fact 11.(Hsiang-Lawson,'71) Homogeneous hypersurfaces in S^n are given by isotropy orbits of rank two symmetric spaces.

Let G/K be a rank two symmetric space, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition. Extend the action $K \curvearrowright \mathfrak{p}$ naturally to $T\mathfrak{p}$:

$$k \cdot (x,Y) = (\mathrm{Ad}k(x), \mathrm{Ad}k(Y)), \quad (x,Y) \in T\mathfrak{p},\ k \in K.$$

Since $\mathfrak{p} \cong \mathbb{R}^n$, we can apply above argument.

Proposition 6. (M.) *Let G/K be a rank two symmetric space, then $U(1) \times K \curvearrowright T\mathfrak{p}$ is a Hamiltonian action with moment map $\mu_0 + \mu : T\mathfrak{p} \to \mathfrak{u}(1)^* \oplus \mathfrak{k}^*$ given by*

$$\begin{aligned} \mu_0(x, Y) &= \frac{1}{2}(\|x\|^2 + \|Y\|^2)\eta, \\ \mu(x, Y) &= -\mathrm{ad}x(Y), \quad (x, Y) \in T\mathfrak{p}. \end{aligned}$$

Corollary 3.5. *When G/K is Hermitian symmetric space, then for $\mathfrak{z} \in \mathfrak{c} \subset \mathfrak{k}$ such that $J = \mathrm{ad}\mathfrak{z}$?Cwe have*

$$\mu(x, \frac{1}{2}Jx) = \mu^H(x) = \frac{1}{2}(\mathrm{ad}x)^2\mathfrak{z}.$$

Remark 3.6. *Proposition 6 holds not only for $g = 4$ but also for* **all the homogeneous cases**.

In our case, $G/K = SO(5) \times SO(5)/SO(5)$ $((m_1, m_2) = (2, 2))$, and $SO(10)/U(5)$ $((m_1, m_2) = (4, 5))$ occur. Let E_{ij} be the 5×5 matrix with 1 in the (i, j) component, and all others 0, and put $G_{ij} = E_{ij} - E_{ji} \in \mathfrak{o}(5) \subset \mathfrak{u}(5)$, $1 \leq i < j \leq 5$.

Theorem 3. (M. to appear in Math. Ann.) *For non-OT-FKM type isoparametric hypersurfaces with $(m_1, m_2) = (2, 2)$, $(4, 5)$, using $\tau = G_{25} + G_{45} \in \mathfrak{k}$, and an element H of the maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$, put $Y_H = [H, \tau] \in \mathfrak{p}$, and extend it to a vector field Y_x via the action of K. Restricting the moment map $\mu_0 + \mu$ of the $U(1) \times K$ action to $C = \{(x, Y_x) = \mathrm{Ad}k(H, Y_H)\} \subset T\mathfrak{p}$, we obtain*

$$F(x) = p\|\mu_0(x, Y_x)\|^2 - q\|\mu(x, Y_x)\|^2$$

Here, when $(m_1, m_2) = (2, 2)$, we have $(p, q) = (3, 4)$, and when $(m_1, m_2) = (4, 5)$, we have $(p, q) = (\frac{3}{4}, 1)$.

Conclusion.

The Cartan-Münzner polynomial of $g = 4$ can be expressed by the square norm of the moment map of certain group action on $T\mathbb{R}^{2l}$ restricted to a hal dimensional submanifold. It is done in both **homogenous and non-homogeneous cases in a unified way**.

Other recent results and future problems.

Fact 12. (Tang-Yan, 2012) *Yau's conjecture on the first eigenvalue of a minimal hypersurface in S^n is affirmative for all the minimal isoparametric hypersurfaces.*

As for transnormal functions which are weakened from isoparametric function, we obtain :

> **Theorem 4. (M. to appear in DGA)** *We call f a transnormal function if it satisfies only the condition (I) of isoparametric function. Then it follows:*
>
> (1) *A complete Riemannian manifold with transnormal function is either diffeomorphic to a vector bundle or a union of two disk bundles.*
>
> (2) *A singular level set of a transnormal function is austere, and so minimal.*

This fact implies that the condition (I) is rather essential.

Remark 3.7. (1) *Since $T\mathbb{R}^n \cong \mathbb{C}^n$, if we complexify the Cartan-Münzner polynomial as a homogeneous complex polynomial, what are complex level sets of $F(z)$ which are hypersurfaces in $\mathbb{C}^{n+1}$ or in $\mathbb{C}P^n$?*

When $g = 3$, it is called the Severi variety. What follows when $g = 4, 6$?

(2) *Z.Z. Tang is studying isoparametric hypersurfaces from various points of view in geometry. For instance, Chern's conjecture, Yau's conjecture, Gromov-Lawson-Schoen-Yau theory on manifold with positive scalar curvature, exotic spheres, and Willmore submanifolds etc. See ArXiv.*

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